Ma 635. Real Analysis I. Lecture Notes

VII. TOPOLOGICAL SPACES and BOREL σ-ALGEBRA

7.1 **Definition** System of subsets τ of set M is *topology* (a system of open sets) if
(1) $\emptyset \in \tau$, $M \in \tau$;
(2) any union of subsets from τ belongs to τ;
(3) any *finite* intersection of the sets from τ also belongs to τ.

7.2 The set of all subsets of M is a topology (it is called as *discrete topology*).

7.3 $\tau = \{\emptyset, M\}$ is a topology.

7.4 System of subsets of of a metric space, which contains the empty set and all open subsets, is a topology. This topology is *generated* by the metric.

7.5 **Definition.** Complements of open sets are called closed sets.

7.6 Empty set and all space M are simultaneously open and closed in the topological space (M, τ).

7.7 Any finite collection of points in a metric space is a closed set.

7.8 The set $\mathbf{Q}$ of rational numbers on the numerical line is neither open nor closed.

7.9 The set $\mathbf{Q}$ on the numerical line is
(a) a countable union of closed sets.
(b) a countable intersection of open sets.

7.10 **Definition.** A collection of sets is called *algebra* if it is closed with respect to finite number of unions, intersections, and complements.

7.11 **Definition.** An algebra is σ-*algebra* if it is closed with respect to *countable* number of unions and intersections.

7.12 **Definition.** Borel sets of topological space (M, τ) form the smallest σ-algebra of sets from M, which contains all open sets.
7.13 The sets of rational and irrational numbers on the numerical line are Borel.

7.14 **Definition.** A function f defined on topological space (M, τ_1) with codomain (M_1, τ_2) is called *Borel function* if the pre-image of every Borel set in M_1 is a Borel set in M.

7.15 A function $f: M \mapsto R$ is Borel when and only when for any half-infinite interval $(-\infty, x)$ its pre-image $f^{-1}(-\infty, x)$ is a Borel set.

7.16 Superposition of Borel functions is also Borel function.

7.17 If f and g are Borel functions then the following functions

$$f + g, \quad cf, \quad f \cdot g, \quad \max\{f, g\}, \quad \min\{f, g\}, \quad |f|, \quad 1/f.$$

are also Borel.

7.18 If every numerical function f_n is Borel and $f_n \to f$ pointwise then the limit function f is also Borel function.

7.19 Set A is Borel if and only if its characteristic function

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

is also Borel.

7.20 Every continuous function is Borel.